THE RANGE OF ANGLES IN THE
EULER-GERGONNE-SODDY TRIANGLE

An Abstract of
a Thesis
Submitted
in Partial Fulfillment
of the Requirement for the Degree
Master of Arts

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University of Northern Iowa
May 2006
ABSTRACT

For any triangle one can construct its Euler-Gergonne-Soddy triangle (EGST). In 1996 it was shown that the EGST is always a right triangle. In this work we compute the range of the other angles of the EGST by calculating the maximum angle of the EGST at the de Longchamps point.
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CHAPTER 1
INTRODUCTION

Apollonius’ problem states given three objects, each of which may be a point, line, or circle, construct a circle which contains all points and is tangent to the lines and circles. The case when all three objects are circles yields up to eight distinct solution circles and was the main study of (Gisch and Ribando, 2004). Of the arrangements of three given circles, the case resulting in only two solution circles occurs when all three circles are mutually tangent as seen in Figure 1.1. This special case is also known as the “kissing coins” or “four coins” problem and it is where the investigation for (Gisch and Ribando, 2004) began.

Figure 1.1: “Kissing coins” and solution circles.

Research into the “kissing coins” problem led immediately to an article by Oldknow (Oldknow, 1996). Let A, B and C be the centers of the three “kissing coins.” Oldknow approaches this special case of Apollonius’ problem by analyzing geometric properties of the Euler-Gergonne-Soddy triangle (EGST) of \( \triangle ABC \). In doing so, Oldknow shows that the centers of the two solution circles are triangle centers for \( \triangle ABC \), lie on the Soddy line, and form multiple harmonic ranges with other triangle centers on the Soddy line. Prior to reading Oldknow’s article I knew little more than the ancient Greek geometers beyond the four triangle centers: the centroid, circumcenter, incenter and orthocenter.
It was not until many years later, beginning with the Fermat point (c. 1650), that other triangle centers were discovered. Again, after this burst of discovery, the topic was relatively unexplored until the 20th century. Its revival can be credited to a handful of individuals, most notably Clark Kimberling of the University of Illinois-Evansville who has cataloged over 3000 triangle centers.

In a forward to Clark Kimberling’s text (Kimberling, 1998), Douglas Hofstadter writes,

A few years ago, I got unexpectedly sucked into an intellectual vortex that has still got me spinning wildly: a passion for something called ‘the elementary geometry of the triangle.’

Hofstadter was referencing Kimberling’s extensive catalog of triangle centers when referring to the elementary geometry of the triangle. Oldknow’s article introduced me to the world of triangle centers and I, too, was immediately sucked in. In fact, I was so enamored by Oldknow’s results that a brief treatment of his work in relation to Apollonius’ problem seemed necessary to include in (Gisch and Ribando, 2004). Using the modern advantage of dynamic geometry software, such as Geometer’s SketchpadTM, we made the observation that the EGST always appears to be long and skinny. Thus, we proposed in (Gisch and Ribando, 2004) that further study could be devoted to finding the range of the angles of the EGST.
CHAPTER 2
THE EULER-GERGONNE-SODDY TRIANGLE

Let us properly define the Euler-Gergonne-Soddy triangle. Given a fixed triangle, which we will often refer to as the reference triangle, we can construct the vertices of the EGST of the reference triangle by intersecting its Euler, Gergonne and Soddy lines. For practical purposes, from this point forward we will often refer to the reference triangle as \(\triangle ABC\), labeled in the usual manner: vertices \(A, B, C\); angles \(A, B, C\); and sidelengths \(a, b, c\) with \(a = |BC|, b = |CA|, c = |AB|\).

We begin by constructing the Euler line. The \textit{Euler line} is the line on which the reference triangle’s orthocenter \(H\), centroid \(G\), circumcenter \(O\), de Longchamps point \(L\), nine-point center \(N\), and a number of other prominent triangle centers lie. With compass and straightedge it is easiest to construct the Euler line as the line containing the \textit{centroid}, the intersection of the medians, and the \textit{orthocenter}, which is obtained by intersecting the altitudes of the reference triangle. The \textit{de Longchamps point} can be defined as the reflection of the orthocenter about the \textit{circumcenter}, obtained as the intersection of the perpendicular bisectors, and is the point of intersection of the Euler and Soddy lines (Court, 1926; Oldknow, 1996).

The \textit{Soddy line} is the line that passes through the inner and outer Soddy centers \(S_i\) and \(S_o\), respectively. To construct these centers take the vertices of the reference triangle and center a circle at each vertex so that three circles are mutually tangent. As mentioned previously, this arrangement of circles is a special case of Apollonius’ problem often referred to as the “kissing coins” problem. For this special case there exist exactly two distinct solution circles that are tangent to the three “kissing coins.” These two circles are called the \textit{inner} and \textit{outer Soddy} circles, and their centers are the \textit{inner} \(S_i\) and \textit{outer Soddy} \(S_o\) centers, respectively (see Figure 2.1). All of these elements are named after Frederick Soddy, the Nobel Prize winner in chemistry, who popularized them with his poem “The Kiss Precise” (Soddy, 1926). The Soddy line also
contains the incenter $I$, Gergonne point $Ge$ and de Longchamps point $L$ among other triangle centers. The *incenter* is the center of the *incircle*, the inscribed circle of the reference triangle, and is obtained by intersecting the angle bisectors. The three points of tangency of the incircle and reference triangle define the *contact triangle*. Though we defined the Soddy line by the Soddy centers it is most easily constructed by creating the line through the incenter and *Gergonne point*, the perspector or perspective center of the reference triangle and contact triangle. The *perspector* for any two perspective triangles is the point of concurrence of the three lines connecting the corresponding vertices of the two triangles.

![Figure 2.1: Soddy centers for $\triangle ABC$.](image)

Of the three lines forming the EGST, the Gergonne line is the most complicated to construct as it involves the most construction steps. Just as the Gergonne point is the perspector of the reference triangle and its contact triangle, the Gergonne line is the perspectrix of these same two triangles. In general, the *perspectrix* of any two perspective triangles is the line joining the three collinear points of intersection of the extensions of corresponding sides. In the specific case of the reference triangle and its contact triangle, if one extends the sides of the reference triangle and contact triangle they meet pairwise in three points known as the *Nobbs points* (Oldknow, 1996). The Nobbs points $N_1$, $N_2$ and $N_3$ are collinear and form the Gergonne line as illustrated in
Figure 2.2. In addition to the Nobbs points, the Gergonne line contains the *Fletcher point* $Fl$ and *Evans point* $Ev$ which are defined as the intersection of the Gergonne line with the Euler and Soddy lines, respectively. Most significantly it appears that Oldknow was the first to prove, as recently as 1996, that the Soddy line and Gergonne line are always perpendicular (Oldknow, 1996; Weisstein n.d.a).

![Gergonne Line](image)

Figure 2.2: The Gergonne line for $\triangle ABC$.

Therefore, the intersections of the Euler, Gergonne and Soddy lines create the vertices $L$, $Ev$ and $Fl$ of the EGST of reference triangle $\triangle ABC$ as shown in Figure 2.3. As with the reference triangle we will use the labels for the vertices of the EGST and the corresponding angles at its vertices interchangeably. Thus, we reiterate that the EGST always possesses a right angle at the Fletcher point; equivalently, the angle $Fl$ is always a right angle.

Degenerate reference triangles and degenerate EGST’s play a central role in this thesis. To discuss these occurrences it is necessary to extend the plane model to include a *line at infinity*. Intuitively we might believe that degenerate triangles do not have triangle centers and therefore do not have a corresponding EGST. Surprisingly, this is not the case.

For the case of a degenerate triangle $\triangle ABC$ of three distinct points, we can think of any triangle center of $\triangle ABC$ as the limiting point as the altitude from $C$ to $\overrightarrow{AB}$ goes to zero. When all three points of $\triangle ABC$ lie on the line $\overrightarrow{AB}$, the altitudes are the three lines perpendicular to $\overrightarrow{AB}$ at their respective vertices. This implies that the orthocenter
lies at the point at infinity in the direction along which \( C \) approaches \( \overrightarrow{AB} \). Also, the centroid becomes the projection of itself onto \( \overrightarrow{AB} \) (see Figure 2.4). The Euler line is then a line perpendicular to \( \overrightarrow{AB} \) at the centroid. Likewise, the limiting points of the inner Soddy center and incenter coincide with \( C \) and the outer Soddy center becomes the point at infinity in the opposite direction that \( C \) approaches \( \overrightarrow{AB} \). The Soddy line then becomes the perpendicular line to \( \overrightarrow{AB} \) through the point \( C \). We then see that for degenerate triangles of this form the Euler and Soddy lines are parallel, as illustrated in the right of Figure 2.4.

Figure 2.3: The Euler-Gergone-Soddy triangle of \( \triangle ABC \).

Figure 2.4: Degenerate triangles of non-coincident points.
We must also consider degenerate triangles where exactly two points coincide. In this case of degenerate \( \triangle ABC \), suppose \( A \) and \( C \) coincide. Then if we fix the angle at \( A \) we can think of any triangle center as the limiting point as the angle at \( B \) goes to zero, as illustrated in the left of Figure 2.5. Also, the Fletcher point and incenter coincide with \( A \) and the Gergonne and Soddy lines therefore intersect at \( A \), as depicted in the right of Figure 2.5. Notice that we have a choice for the fixed angle of \( A \) satisfying \( 0 \leq m\angle A \leq \pi \). Thus, for this second case of coincidental points it is necessary to describe the degenerate triangle by its angles to distinguish amongst degenerate triangles of this form. For example, the degenerate \( \triangle ABC \) where \( A \) and \( C \) coincide such that \( m\angle A = 30^\circ \) and \( m\angle B = 0^\circ \) is different from the case when \( m\angle A = 90^\circ \) and \( m\angle B = 0^\circ \) when analyzing their triangle centers.

For the case of isosceles triangles \( \triangle ABC \), the EGST is degenerate since its Euler and Soddy lines coincide with the axis of symmetry for \( \triangle ABC \) (Oldknow, 1996). This implies that the EGST is a degenerate triangle where \( Ev \) and \( Fl \) coincide. If the triangle is equilateral there is no unique axis of symmetry and the points \( G, Ge, H, L, N, O, S_i, \) and \( S_o \) all coincide with \( I \). Thus, the Euler and Soddy lines do not exist as real lines. However, the Gergonne line becomes the line at infinity (Oldknow, 1996).
Therefore, for the case of equilateral triangles, the Euler and Soddy lines coincide and can be chosen as any line passing through the incenter.

We now see that the angle at $L$ has a minimum value of zero degrees for the case of isosceles triangles. Since the EGST always has a right triangle at $Fl$, making the angles at $L$ and $Ev$ complementary, it is sufficient to study the range of the angle at $L$. Thus, in order to compute the possible angles of the EGST, it only remains to find the maximum for the angle at $L$. Chapters 3 and 4 of this thesis are devoted to computing the maximum angle at $L$. We conclude with Chapter 5 where we analyze right angled reference triangles.
CHAPTER 3
COMPUTING THE ANGLE AT THE DE LONGCHAMPS POINT

We now have the foundation to achieve the goal of this paper and realize that it is necessary to complete the following steps: we must generate all triangles in the plane and for each triangle compute the angle between its Euler and Soddy lines.

In the proceeding proposition we drastically reduce the task of generating all triangles in the plane with the aid of trilinear coordinates. As Kimberling points out in (Kimberling, 1998),

[Trilinear coordinates] may be compared to the giant step forward, in the history of science, known as cartesian coordinates. In fact, H. S. M. Coxeter has written, “Möbius’s invention of homogeneous coordinates was one of the most far-reaching ideas in the history of mathematics.”

In the proof of the proposition we make use of the fact that triangle centers are homogeneous in their coordinates. We strongly suggest that one visit (Kimberling, 1994, 1998; Oldknow, 1996) for a more insightful review of trilinear coordinates and center-functions to fully appreciate their power and beauty.

**Proposition 3.1.** The EGST’s of similar triangles are similar.

**Proof.** By definition triangle centers are defined by homogeneous trilinear coordinates. Homogeneity ensures that similar triangles have similarly situated triangle centers (Kimberling, 1998, p. 164). Thus, dilating $\triangle ABC$ by a factor of $k$ does not change the relative location of its triangle centers $Ev$, $Fl$, and $L$. Therefore, the image of the EGST of $\triangle ABC$ under a similarity transformation is similar to its preimage. 

By Proposition 3.1 we now see that it is only necessary to generate all triangles up to similarity. The following proposition accomplishes this task.

**Proposition 3.2.** There is a bijective correspondence between the similarity class representatives of triangles and the points in the domain $\mathcal{D}$ given as

$$\mathcal{D} = \left\{ (\alpha, \beta) : 0 \leq \alpha \leq \beta \leq \frac{\pi}{2} - \frac{\alpha}{2} \right\}.$$
Proof. \((\Rightarrow)\) Given a triangle we need to find a corresponding pair \((\alpha, \beta) \in \mathcal{D}\). Up to similarity, a triangle is determined by its three angles. Given an arbitrary triangle \(\triangle ABC\) label its angles \(\alpha, \beta\) and \(\gamma\) ordered so that \(\alpha \leq \beta \leq \gamma\). Then a similarity class representative for \(\triangle ABC\) depends upon \(\alpha\) and \(\beta\) only as \(\gamma = \pi - \alpha - \beta\). It remains to show that \(\beta \leq \frac{\pi}{2} - \frac{\alpha}{2}\). It follows from \(\alpha + \beta + \gamma = \pi\) and \(\beta \leq \gamma\) that \(\alpha + 2\beta \leq \pi\) and therefore \(\beta \leq \frac{\pi}{2} - \frac{\alpha}{2}\).

\((\Leftarrow)\) We need to show that for any pair \((\alpha, \beta) \in \mathcal{D}\) there exists a triangle with angles \(\alpha, \beta\) and \(\gamma = \pi - \alpha - \beta\) so that \(\beta \leq \gamma\). Since \((\alpha, \beta) \in \mathcal{D}\) we know that \(\beta \leq \frac{\pi}{2} - \frac{\alpha}{2}\), which implies that \(2\beta \leq \pi - \alpha\). Since

\[
\gamma = (\pi - \alpha) - \beta \geq 2\beta - \beta = \beta,
\]

we have satisfied the needed conditions.

Therefore there is a bijective correspondence between the similarity class representatives of triangles and the points in the domain \(\mathcal{D}\), which is illustrated in Figure 3.1.

Now, for each triangle in \(\mathcal{D}\) we need to calculate the angle between the Euler and Soddy lines. We proceed by representing the Euler and Soddy lines by vectors \(\vec{u}\) and \(\vec{v}\), respectively. In doing so, to compute the angle \(\theta\) between the Euler line and Soddy line at the de Longchamps point we only need to compute the angle between the vectors \(\vec{u}\) and \(\vec{v}\) using the dot product formula

\[
\theta = \arccos \left( \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \right).
\tag{3.1}
\]

Thus, we need to calculate \(\vec{u}\) and \(\vec{v}\) for each point \((\alpha, \beta) \in \mathcal{D}\).

Recall that the Euler line contains the circumcenter \(O\) and the de Longchamps point \(L\). Likewise, the Soddy line contains \(L\) and the incenter \(I\). Thus, the Euler and Soddy lines may be represented as vectors \(\vec{u} = \overrightarrow{OL}\) and \(\vec{v} = \overrightarrow{IL}\), respectively.
The representations for $\vec{u}$ and $\vec{v}$ and their included angle will depend on a choice of embedding for the reference triangle and this necessitates the following definition.

**Definition 1.** A family of triangles is a choice of vertex embeddings in $\mathbb{R}^2$ so that every similarity class of triangles is represented.

Notice that this definition allows for similarity classes to be represented more than once. In Chapter 5 we exploit the idea that a different family of triangles may better allow us to perform particular calculations.

In an effort to create a family of triangles, let $A = (1, 0)$ and $B = (0, 0)$. We define $l_A$ as the line through $A$ creating angle $\alpha$, measured clockwise from the $x$-axis. Similarly, we define $l_B$ as the line through $B$ creating angle $\beta$, measured counter-clockwise from the $x$-axis. Let $C = l_A \cap l_B$, and call the angle created at its intersection $\gamma$. Then $\triangle ABC$ corresponds to the point $(\alpha, \beta)$ in the domain $D$. It remains for us to calculate the coordinates of vertex $C$ given $\alpha$ and $\beta$. 
**Proposition 3.3.** For \((\alpha, \beta) \in \mathcal{D}\) the coordinates of the point \(C = l_A \cap l_B\) are

\[
C = \left( \frac{\sin \alpha \cos \beta}{\sin(\alpha + \beta)}, \frac{\sin \alpha \sin \beta}{\sin(\alpha + \beta)} \right).
\]

**Proof.** As usual, let \(a\) and \(b\) be the sides opposite vertices \(A\) and \(B\), respectively. Let \(|CP| = h\) where \(P\) is the foot of the altitude from vertex \(C\). Then \(|AP| = b \cos \alpha\) and \(|BP| = a \cos \beta\) as shown in Figure 3.2. Using the Pythagorean theorem and trigonometric angle addition formulas results in

\[
a = \frac{\sin \alpha}{\sin(\alpha + \beta)} \quad \text{and} \quad b = \frac{\sin \beta}{\sin(\alpha + \beta)}.
\]

Then, as \(C = (a \cos \beta, a \sin \beta)\), substituting \(a\) and \(b\) gives rise to

\[
C = \left( \frac{\sin \alpha \cos \beta}{\sin(\alpha + \beta)}, \frac{\sin \alpha \sin \beta}{\sin(\alpha + \beta)} \right).
\] (3.2)

\[
\square
\]

**Definition 2.** The collection \(\mathcal{H}\) of triangles \(\triangle ABC\) with coordinates

\[
A = (1, 0), \quad B = (0, 0) \quad \text{and} \quad C = \left( \frac{\sin \alpha \cos \beta}{\sin(\alpha + \beta)}, \frac{\sin \alpha \sin \beta}{\sin(\alpha + \beta)} \right)
\]

where \((\alpha, \beta) \in \mathcal{D}\) is defined as the Unit Hypotenuse family.
We note that the use of the word hypotenuse is an abuse of its meaning since it only applies to right triangles. However, it reminds us that in general the longest side of any triangle in \( \mathcal{H} \) has fixed length one. Using the Unit Hypotenuse family the formulas for the needed centers can then be calculated and are shown in Table 3.1. With the exception of the de Longchamps point, we calculated each of these centers using formulas from Roman Calvet’s text (Calvet, 2000, pp. 68-77). Recall that the de Longchamps point is the reflection of the orthocenter about the circumcenter. Thus, using parametric equations we have \( L = 2O - H \), giving us the expression listed in Table 3.1. We also mention that we choose expressions for the centers in Table 3.1 so that each coordinate would be symmetric with \( \alpha \) and \( \beta \) whenever possible.

<table>
<thead>
<tr>
<th>Triangle Center</th>
<th>Symbol</th>
<th>Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circumcenter</td>
<td>( O )</td>
<td>( \left( \frac{1}{2}, \frac{-\cos(\alpha + \beta)}{2 \sin(\alpha + \beta)} \right) )</td>
</tr>
<tr>
<td>Orthocenter</td>
<td>( H )</td>
<td>( \left( \frac{\sin \alpha \cos \beta}{\sin(\alpha + \beta)}, \frac{\cos \alpha \cos \beta}{\sin(\alpha + \beta)} \right) )</td>
</tr>
<tr>
<td>Incenter</td>
<td>( I )</td>
<td>( \left( \frac{\sin \alpha + \sin \alpha \cos \beta}{\sin(\alpha + \beta) + \sin \alpha + \sin \beta}, \frac{\sin \alpha \sin \beta}{\sin(\alpha + \beta) + \sin \alpha + \sin \beta} \right) )</td>
</tr>
<tr>
<td>de Longchamps point</td>
<td>( L )</td>
<td>( \left( \frac{\cos \alpha \sin \beta}{\sin(\alpha + \beta)}, \frac{-\cos(\alpha + \beta) - \cos \alpha \cos \beta}{\sin(\alpha + \beta)} \right) )</td>
</tr>
</tbody>
</table>

Table 3.1: Triangle centers for \((\alpha, \beta) \in \mathcal{D}\) using the Unit Hypotenuse family \( \mathcal{H} \).

With the coordinates for the needed triangle centers in our possession we can compute vectors \( \vec{u} \) and \( \vec{v} \):

\[
\vec{u} = \left\langle \frac{\sin(\beta - \alpha)}{2 \sin(\alpha + \beta)}, \frac{-\cos(\alpha + \beta) - 2 \cos \alpha \cos \beta}{2 \sin(\alpha + \beta)} \right\rangle
\]

\[
\vec{v} = \left\langle \frac{\sin(\beta - \alpha) + \sin \beta - \sin \alpha}{2 \sin(\alpha + \beta)}, \frac{-\cos(\alpha + \beta) - 2 \cos \alpha \cos \beta - \cos \alpha - \cos \beta + 1}{2 \sin(\alpha + \beta)} \right\rangle.
\]

Thus, using (3.1), for any triangle in \( \mathcal{D} \) the measure of the angle at \( L \) is given by the
function $f : \mathcal{D} \rightarrow \mathbb{R}$ defined by

$$f(\alpha, \beta) = \arccos \left( \frac{\lambda}{\mu} \right)$$

where $\lambda$ and $\mu$ are defined in the following proposition.

**Proposition 3.4.** For any given triangle whose two smaller angles are $\alpha$ and $\beta$ the angle of the EGST at the de Longchamps point $L$ is given by $f(\alpha, \beta) = \arccos \left( \frac{\lambda}{\mu} \right)$ where

$$\lambda = \cos(\alpha + \beta) \cdot (8 \cos \alpha \cos \beta + 2 \cos \beta + 2 \cos \alpha + 1) - 4 \cos \alpha \cos \beta + \cos \beta + \cos \alpha + 1$$

$$\mu = \left( \sqrt{\sin^2(\beta - \alpha) + (-2 \cos(\alpha + \beta) - \cos(\beta - \alpha))^2} \right) \sqrt{(\sin(\beta - \alpha) + \sin \beta - \sin \alpha)^2 + (1 - 2 \cos(\alpha + \beta) - \cos(\beta - \alpha) - \cos \alpha - \cos \beta)^2}.$$ 

As the Gergonne and Soddy lines always form a right angle at $Fl$ this implies that

$$f(\alpha, \beta) = \arccos \left( \frac{\lambda}{\mu} \right) \leq \frac{\pi}{2}.$$ 

Thus, $0 \leq \frac{\lambda}{\mu} \leq 1$. Also, as $\arccos(x)$ is decreasing on the interval $[0, 1]$ the function $f$ has a maximum where the function $F : \mathcal{D} \rightarrow \mathbb{R}$ defined by

$$F(\alpha, \beta) = \frac{\lambda}{\mu}$$

has a minimum. Therefore, $f$ achieves its maximum on $\mathcal{D}$ where $F$ achieves its minimum.
CHAPTER 4
ANALYSIS OF THE FUNCTIONS $f$ AND $F$

In order to calculate the maximum angle at the de Longchamps point, we need to analyze the functions

$$f(\alpha, \beta) = \arccos \left( \frac{\lambda}{\mu} \right) \quad \text{and} \quad F(\alpha, \beta) = \frac{\lambda}{\mu}$$

on the domain $D$, where $\frac{\lambda}{\mu}$ is defined in Proposition 3.4.

**Proposition 4.1.** The function $f$ achieves a minimum value of zero and a maximum value of $\arccos \left( \frac{2\sqrt{6}}{5} \right)$ on the boundary of $D$.

**Proof.** We analyze the boundary of $D$ separately along the three segments $\beta = \alpha$, $\beta = \frac{\pi}{2} - \frac{\alpha}{2}$, and $\alpha = 0$. Let us first examine the boundary where $\beta = \alpha$. If we substitute $\beta = \alpha$ into our function $F$ and perform successive trigonometric simplifications, we have

$$F(\beta, \beta) = 1.$$

Thus, along the boundary where $\beta = \alpha$ the value of the angle of the EGST at $L$ is

$$f(\beta, \beta) = \arccos(1) = 0.$$

Looking at the boundary $\beta = \frac{\pi}{2} - \frac{\alpha}{2}$, rather than substitute $\beta = \frac{\pi}{2} - \frac{\alpha}{2}$ into our function $F$, let us substitute $\alpha = \pi - 2\beta$. In doing so we obtain

$$F(\pi - 2\beta, \beta) = 1.$$

Thus, along the boundary where $\beta = \frac{\pi}{2} - \frac{\alpha}{2}$ the value of the angle of the EGST at $L$ is

$$f(\pi - 2\beta, \beta) = \arccos(1) = 0.$$
We point out that the last two results are not surprising. These two boundaries represent the isosceles triangles $\beta = \alpha$ and $\beta = \gamma$. As noted earlier in Chapter 2, the Euler and Soddy lines coincide for isosceles triangles, so the angle at the de Longchamps point vanishes.

We finish by analyzing the boundary $\alpha = 0$. If we fix $\alpha = 0$ the function $F$ becomes the single variable function

$$F(0, \beta) = \frac{1 + 5 \cos^2 \beta}{\sqrt{1 + 8 \cos^2 \beta} \sqrt{1 + 3 \cos^2 \beta}}.$$

Taking the derivative we obtain

$$F'(0, \beta) = -\frac{\cos \beta \sin \beta (7 \cos^2 \beta - 1)}{(1 + 8 \cos^2 \beta)^{3/2}(1 + 3 \cos^2 \beta)^{3/2}}.$$

Setting the derivative equal to zero we have

$$\cos \beta \sin \beta (7 \cos^2 \beta - 1) = 0.$$

Thus, $F(0, \beta)$ has critical points in $\{0, \arccos \left(\frac{1}{\sqrt{7}}\right), \frac{\pi}{2}\}$ along the boundary $\alpha = 0$. Evaluating $F$ at each of these points we have

$$F(0, 0) = 1$$
$$F\left(0, \arccos \left(\frac{1}{\sqrt{7}}\right)\right) = \frac{2\sqrt{6}}{5} \approx .9798$$
$$F\left(0, \frac{\pi}{2}\right) = 1.$$

Thus, along the boundary $\alpha = 0$, the minimum value of $F$ is $\frac{2\sqrt{6}}{5}$ and the maximum is 1.

Therefore, along the boundary of the domain $\mathcal{D}$, $f$ has a maximum value of $\arccos \left(\frac{2\sqrt{6}}{5}\right)$ and minimum of zero as shown in Figure 4.1.
Theorem 4.2. Provided that $\frac{\partial F}{\partial \alpha}$ is positive for all $(\alpha, \beta) \in \mathcal{D}$, the range of the angle of the Euler-Gergonne-Soddy triangle at the de Longchamps point $L$ is

$$0 \leq m\angle L \leq \arccos\left(\frac{2\sqrt{6}}{5}\right) \approx 11.5370^\circ.$$ 

Proof. Under the hypothesis that $\frac{\partial F}{\partial \alpha} > 0$ for all $(\alpha, \beta) \in \mathcal{D}$, $F$ has no critical points in the interior of the domain $\mathcal{D}$. Thus, the maximum and minimum of $F$ occur along the boundaries of $\mathcal{D}$ as analyzed in Proposition 4.1. Therefore, the range of the angle of the Euler-Gergonne-Soddy triangle at the de Longchamps point $L$ is

$$0 \leq m\angle L \leq \arccos\left(\frac{2\sqrt{6}}{5}\right) \approx 11.5370^\circ.$$ 

The assumption that there are no critical points in the interior seems rather hopeful at first. At this time analysis of the partial of $F$ with respect to $\alpha$ could not be shown to support this fact. Since $F$ invokes a number of rational expressions of trigonometric functions of $\alpha$ and $\beta$, so too does the derivative, and combined with the fact that the
values of $F$ over $\mathcal{D}$ range from approximately .9798 to 1 give rise to the complications we have encountered. However, the graph of $F$ over the domain $\mathcal{D}$, shown in Figure 4.2, and the contour map of $F$ over the same domain, Figure 4.3, support the assumption in Theorem 4.2. Furthermore, for specific cases of $\beta$ we calculated the derivative of $F$ with respect to $\alpha$ and the result was a positive derivative in each case. Unfortunately, these results could not be extended to prove this fact for arbitrary $\beta$. Thus, with all of this evidence in hand, we feel very certain that the hypothesis $\frac{\partial F}{\partial \alpha} > 0$ is true, but we do recognize its shortcomings.

![Figure 4.2: Graph of $F$ over the domain $\mathcal{D}$.](image)
Figure 4.3: Contour plot of $F$ over $D$. 
CHAPTER 5
THE RANGE OVER RIGHT TRIANGLES

The results in Chapter 4 are striking; however, it should be pointed out that we
explored a number of triangle families before we were able to make these achievements.
Depending upon what one wishes to calculate, certain families of triangles may have
advantages over others. For instance, calculating the range of the angle of the EGST at
L for all right triangles can be done with the Unit Hypotenuse family but results in a
rather crude form of trigonometric terms that can not be reduced. In this chapter we
present a different triangle family and give precise calculations for the range of the angle
of the EGST at L for all right triangles. We note that this also demonstrates that, using
a different parameterization, we can analyze a one-parameter sub-family interior to D.

Proposition 5.1. The collection $\mathcal{R}$ of triangles with coordinates $A(1,0)$, $B(0,0)$ and
$C \in \{(0,y) \mid 0 \leq y \leq 1\}$ generates all right triangles up to similarity; we call it the
Right triangle family.

Proof. Let an arbitrary right triangle $T$ be given. Label its vertices $A$, $B$ and $C$ so
that $B$ is the right angle and $A$ is the smallest angle. Then translate $B$ to the origin.
Next, rotate the triangle so that $\overline{AB}$ lies on the positive $x$-axis. If $C$ lies on the negative
$y$-axis reflect $T$ about the $x$-axis. Finally, dilate $T$ about the origin so that $A = (1,0)$.
The composition of these transformations result in a triangle similar to $T$ since each
individual transformation is itself a similarity transformation. Furthermore, by our
construction $\overline{BC}$ is opposite the smallest angle of our triangle which implies that
$|AC| \leq |AB| = 1$. Thus $C = (0,y)$ where $0 \leq y \leq 1$. Therefore the collection $\mathcal{R}$ generates
all right triangles up to similarity. \hfill \Box

Using the family $\mathcal{R}$ we will use the same approach as in Chapters 3 and 4. That is
to say, we create vectors $\vec{u}$ and $\vec{v}$ and calculate the angle between them using the dot
product formula (3.1). As before, the minimum of $\frac{\lambda}{\mu}$ for all triangles in $\mathcal{R}$ is the desired
maximum of $f$. Proceeding as such we have calculated the triangle center coordinates and listed them in Table 5.1.

Using the centers for $\mathcal{R}$ we have

$$
\vec{u} = (1, y)
$$

$$
\vec{v} = \left\langle \frac{1 + \sqrt{1 + y^2}}{1 + y + \sqrt{1 + y^2}}, \frac{y(y + \sqrt{1 + y^2})}{1 + y + \sqrt{1 + y^2}} \right\rangle.
$$

Notice that we represent the Euler line as the vector $\vec{u} = \overrightarrow{HL}$ rather than $\vec{u} = \overrightarrow{OL}$ as was done in Chapter 3. This was done for the sake of simplifying calculations. Thus, for any triangle in the family $\mathcal{R}$, the measure of the angle at $L$ is given by the function $f : \mathcal{R} \to \mathbb{R}$ defined by

$$
f(y) = \arccos \left( \frac{\sqrt{2(1 + \sqrt{1 + y^2} + y^3 + y^2\sqrt{1 + y^2})}}{2\sqrt{(y^2 - y + 1)(1 + y^2)(y^2 + y + 1 + y\sqrt{1 + y^2} + \sqrt{1 + y^2})}} \right).
$$

**Theorem 5.2.** For the collection of all right triangles the range of the angle of the Euler-Gergonne-Soddy triangle at the de Longchamps point $L$ is

$$
0 \leq m\angle L \leq 5.9244^\circ.
$$
Proof. We analyze $F : \mathbb{R} \to \mathbb{R}$ where

$$F(y) = \left( \frac{\sqrt{2}(1 + \sqrt{1 + y^2} + y^3 + y^2 \sqrt{1 + y^2})}{2\sqrt{(y^2 - y + 1)(1 + y^2)(y^2 + y + 1 + y\sqrt{1 + y^2} + \sqrt{1 + y^2})}} \right).$$

Taking the derivative with respect to $y$ we have

$$F'(y) = \frac{-y^2(y^2 + 1)(3y^4 - 10y^3 + 10y^2 - 10y + 3)}{\phi}$$

where

$$\phi = \left( 4(y^2 - y + 1)(y^2 + 1)^{3/2}(y^2 + y + 1 + (1 + y)\sqrt{y^2 + 1}) \right) \left( \sqrt{y^6 + 2y^4 + 2y^2 + 1 + (y^5 + y^3 + y^2 + 1)\sqrt{y^2 + 1}} \right).$$

We see by inspection that the numerator of $F'$ has roots in $\{0, i, -i\}$. Lemma 5.3 shows that the quartic expression $3y^4 - 10y^3 + 10y^2 - 10y + 3$ has real roots in

$$\left\{ \frac{\sqrt{13} - \sqrt{2 + 10\sqrt{13}}}{6}, \frac{\sqrt{13} + \sqrt{2 + 10\sqrt{13}}}{6} + \frac{5}{6} \right\}.$$

Hence, the critical points of $F(y)$ include the endpoints of the interval $[0, 1]$ and only one of the real roots of the quartic:

$$\left\{ 0, \frac{\sqrt{13} - \sqrt{2 + 10\sqrt{13}}}{6} + \frac{5}{6}, 1 \right\}.$$

Evaluating $F$ at each of these real roots we have,
\[ F(0) = 1, \]
\[ F\left(\frac{\sqrt{13} - \sqrt{2 + 10\sqrt{13}}}{6} + \frac{5}{6}\right) \approx 0.9947 \quad \text{and} \]
\[ F(1) = 1. \]

Thus, \( F \) achieves a minimum at \( \frac{\sqrt{13} - \sqrt{2 + 10\sqrt{13}}}{6} + \frac{5}{6} \) and maximum at \( y = 0 \) and \( y = 1 \).

This implies that \( f \) achieves its maximum at \( \frac{\sqrt{13} - \sqrt{2 + 10\sqrt{13}}}{6} + \frac{5}{6} \), given as

\[ f\left(\frac{\sqrt{13} - \sqrt{2 + 10\sqrt{13}}}{6} + \frac{5}{6}\right) \approx 5.9244^\circ. \]

Thus, for the collection of right triangles the range of the angle of the Euler-Gergonne-Soddy triangle at the de Longchamps point \( L \) is

\[ 0 \leq m\angle L \leq 5.9244^\circ. \]

\[ \square \]

Theorem 5.2 showed that we need to find the roots of the quartic expression

\[ 3y^4 - 10y^3 + 10y^2 - 10y + 3. \]

Ferrari was the first to develop an algebraic technique for solving the general quartic, which was stolen and published in Cardano’s *Ars Magna* in 1545 (Boyer and Merzbach, 1991). In the proceeding lemma we use Ferrari’s technique as described in (Weisstein, n.d.b) to find the roots of the expression

\[ 3y^4 - 10y^3 + 10y^2 - 10y + 3. \]

**Lemma 5.3.** The quartic expression \( 3y^4 - 10y^3 + 10y^2 - 10y + 3 \) has real roots in

\[ \left\{ \frac{\sqrt{13} - \sqrt{2 + 10\sqrt{13}}}{6} + \frac{5}{6}, \frac{\sqrt{13} + \sqrt{2 + 10\sqrt{13}}}{6} + \frac{5}{6} \right\}. \]
Proof. Before applying Ferrari’s technique the leading coefficient of the quartic must be one. Thus, we have

\[ y^4 - \frac{10}{3} y^3 + \frac{10}{3} y^2 - \frac{10}{3} + 1. \quad (5.1) \]

The first step of Ferrari’s technique is to eliminate the \( y^3 \) term by substituting \( y = \hat{y} - \frac{a_3}{4} \) where \( a_3 \) is the coefficient of the \( y^3 \) term. In doing so we have \( y = \hat{y} + \frac{5}{6} \) and

\[
\left( \hat{y} + \frac{5}{6} \right)^4 - \frac{10}{3} \left( \hat{y} + \frac{5}{6} \right)^3 + \frac{10}{3} \left( \hat{y} + \frac{5}{6} \right)^2 - \frac{10}{3} + 3 \\
= \hat{y}^4 - \frac{5}{6} \hat{y}^2 - \frac{65}{27} \hat{y} - \frac{131}{144}. \quad (5.2)
\]

This quartic can be solved by finding what is called the resolvent cubic. Once the resolvent cubic is found it allows us to write the quartic equation as the difference of two squared terms, which can then be factored into two quadratic expressions. The roots of these quadratic expressions can then be easily solved to yield our solutions.

The resolvent cubic is given by the equation \( q^2 = 4(u - p)(u^2/4 - r) \) where \( p, q \) and \( r \) are the coefficients of the form \( x^4 + px^2 + qx + r \) for equation (5.2). Hence, the resolvent cubic is

\[
-\frac{4225}{729} + 4 \left( u + \frac{5}{6} \right) \left( \frac{u^2}{4} + \frac{131}{144} \right) = -\frac{16115}{5832} + u^3 + \frac{131}{36} u + \frac{5}{6} u^2 \\
= \frac{1}{5832} (18u - 11)(324u^2 + 468u + 1465).
\]

It is only required that we calculate one of the real roots of the resolvent cubic so we easily have \( u = \frac{11}{18} \). Using this solution we can then write the quartic equation (5.2) as the difference of two squared terms \( P^2 - Q^2 \) where

\[
P = \hat{y}^2 + u = \hat{y}^2 + \frac{11}{36} \\
Q = \sqrt{u - p} - \frac{q}{2\sqrt{u - p}} = \frac{(6\hat{y} + 5)\sqrt{13}}{18},
\]

which can be factored as

$$(P - Q)(P + Q) = \left(\hat{y}^2 + \frac{11}{36} - \frac{(6\hat{y} + 5)\sqrt{13}}{18}\right)\left(\hat{y}^2 + \frac{11}{36} + \frac{(6\hat{y} + 5)\sqrt{13}}{18}\right).$$

As Ferrari promised, we have factored (5.2) into quadratic expressions which give us the following roots:

$$\hat{y}_1 = \frac{\sqrt{13} + \sqrt{2 + 10\sqrt{13}}}{6} \quad \hat{y}_2 = \frac{\sqrt{13} - \sqrt{2 + 10\sqrt{13}}}{6} \quad \hat{y}_3 = -\frac{\sqrt{13} + \sqrt{2 - 10\sqrt{13}}}{6} \quad \hat{y}_4 = -\frac{\sqrt{13} - \sqrt{2 - 10\sqrt{13}}}{6}.$$

Note that we only wish to find real solutions of our original quartic leaving $\hat{y}_1$ and $\hat{y}_2$. Also, we began the process by substituting $y = \hat{y} + \frac{5}{6}$. Thus, the real solutions of the original quartic $3y^4 - 10y^3 + 10y^2 - 10y + 3$ are

$$y_1 = \hat{y}_1 + \frac{5}{6} = \frac{\sqrt{13} + \sqrt{2 + 10\sqrt{13}}}{6} + \frac{5}{6} \approx 2.4624$$
$$y_2 = \hat{y}_2 + \frac{5}{6} = \frac{\sqrt{13} - \sqrt{2 + 10\sqrt{13}}}{6} + \frac{5}{6} \approx 0.4061.$$
REFERENCES


